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FORCED CONVECTION HEAT TRANSFER IN A CIRCULAR TUBE WITH NON-UNIFORM HEAT FLUX AROUND THE CIRCUMFERENCE

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Abstract—An analytical solution, based on a simple postulated distribution of the effective thermal conductivity of the fluid, is derived for the temperature variations around the circumference of a circular tube arising from non-uniform wall heat flux. The results are in good agreement with an existing complex numerical solution and in some cases are probably more accurate. The accuracy of both solutions is mainly determined by the accuracy with which the effective eddy diffusivity of heat in the fluid is known. Published experimental data for a non-uniformly heated tube are used to establish the best value. A simple formula for calculating temperature variations around tubes in practical cases is proposed.

NOMENCLATURE

- b, mean tube wall thickness;
- h, heat-transfer coefficient;
- k, fluid molecular thermal conductivity;
- k^1, fluid eddy thermal conductivity;
- k_s, tube wall thermal conductivity;
- q, heat flux from wall to fluid;
- r, radial co-ordinate;
- r_0, tube radius;
- s, boundary-layer thickness;
- t, temperature;
- $\bar{t}_a$, mean temperature difference between tube and fluid for uniform heat flux;
- u, fluid velocity;
- u^*, friction velocity, equation (10);
- ϵ_h, eddy diffusivity of heat;
- ϵ_m, eddy diffusivity of momentum;
- θ, angular co-ordinate;
- φ, phase angle;
- ν, fluid molecular diffusivity of momentum;
- ω, electrical resistivity.

Non-dimensional parameters

- A, eddy diffusivity constant, equation (9);
- B, variation of tube wall thickness;
- C, variation of heat generation rate;
- D, equation (6);
- f, friction factor;
- n, harmonic number;
- N, conduction parameter, equation (27);
- Nu, Nusselt number;
- Pr, Prandtl number;
- Re, Reynolds number;
- S, temperature, equations (2)–(4);
- $\bar{u}^+$, velocity, equation (10);
- $\bar{t}_a^+$, temperature, equation (21).

Subscript numbers indicate amplitudes of harmonics.

INTRODUCTION

THE TEMPERATURE distribution arising from a non-uniform heat flux distribution around the circumference of a circular tube cooled by an internal flow is of considerable practical importance as these heat transfer conditions occur in many types of equipment and local overheating could cause failures.

In principle, the temperature distribution throughout the fluid, and hence the wall temperature distribution, can be calculated if firstly the distribution of the total effective diffusivity of momentum (to calculate the velocity distribution) and secondly the distribution of the effective total diffusivity of heat are known throughout the fluid. Considerable empirical data on diffusivity distributions are available and detailed numerical computations of this type have been made. A solution to this particular problem has been given by Reynolds [1]. An alternative analytical solution is presented to provide a check on some apparent anomalies in Reynolds' results and a simple formula for practical application. The results are also used to re-examine the experimental data of Black and Sparrow [2] to obtain the best value of effective eddy diffusivity of heat.

ANALYSIS

The general problem of forced convection heat transfer in a passage with non-uniform heat flux around its circumference is discussed in [3]. It is shown that accurate estimates of the changes in surface temperature corresponding to changes in heat flux distribution within the fluid can be obtained from a very simplified physical model. This is best demonstrated by using the analysis as an example and the explanation is given after equation (7). The model used considers the fluid flow in two regions, a boundary layer of low velocity with the molecular conductivity of the fluid, k, and a turbulent core of effective mean conductivity $\bar{k}^1$. The boundary layer thickness, s, is small, compared with the tube radius r_0 so that the heat flux through it can be considered constant at the local wall value and the turbulent core radius is effectively r_0. Values of s and $\bar{k}^1$ can be chosen for a particular Reynolds number to give a good approximation to the temperature distribution in the fluid corresponding to any heat flux distribution.

Consider the general circumferential heat flux

$$q = q_0 + q_1 \cos(\theta + \varphi_1) + q_2 \cos(2\theta + \varphi_2) + \ldots + q_n \cos(n\theta + \varphi_n) \quad (1)$$

where q_n is the nth harmonic with phase angle φ_n. The temperature distribution within the fluid due to the nth harmonic of heat flux is made up of a boundary-layer drop of $(s/k)\, q_n \times \cos(n\theta + \varphi_n)$ and a core distribution. Since there is no net heat transfer to the fluid by this harmonic of heat flux, the mean fluid temperature is the same as the mean wall temperature and there is no axial temperature variation arising from this harmonic. The temperature distribution in the core is therefore the solution of the two-dimensional conduction problem for the boundary heat flux, which can readily be shown to be (e.g. see laminar solution in [1]) $(r/r_0)^n\, r_0 q_n \cos(n\theta + \varphi_n)/n\bar{k}^1$

∴ Total surface temperature variation about mean =

$$\frac{r_0 q_n \cos(n\theta + \varphi_n)}{k}\left[\frac{s}{r_0} + \frac{k}{n\bar{k}^1}\right]$$

i.e. the nth harmonic of heat flux produces an in-phase nth harmonic of surface temperature.

$$\therefore\ t = \bar{t}_a + t_1 \cos(\theta + \varphi_1) + t_2 \cos(2\theta + \varphi_2) + \ldots + t_n \cos(n\theta + \varphi_n) + \ldots \quad (2)$$

where

$$t_n = \frac{r_0 q_n}{k}\left[\frac{s}{r_0} + \frac{k}{n\bar{k}^1}\right]$$

or

$$S_n = \frac{t_n k}{q_n r_0} = \frac{s}{r_0} + \frac{k}{n\bar{k}^1} \quad (3)$$

is the non-dimensional temperature used in [1].

The mean heat flux q_0 is convected away by the fluid and the heat flux distribution depends on the velocity distribution within the fluid. Two limiting cases can readily be calculated. Firstly, for very high values of Reynolds number, the fluid velocity can be considered to be uniform in the turbulent core.

Then inward radial flux $= q_0 \dfrac{r}{r_0}$

outer temperature
– temperature at radius r

$$= \frac{r_0 q_0}{2\bar{k}^1}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

outer temperature
– mean coolant temperature

$$= \frac{r_0 q_0}{4\bar{k}^1}$$

∴ "fully absorbed" (see [3]) difference between wall temperature and mean fluid temperature =

$$\bar{t}_a = \frac{r_0 q_0}{k}\left[\frac{s}{r_0} + \frac{k}{4\bar{k}^1}\right]$$

or

$$S_0 = \frac{\bar{t}_a k}{q_0 r_0} = \frac{2}{Nu} = \frac{s}{r_0} + \frac{k}{4\bar{k}^1}. \quad (4)$$

The second limiting case is laminar flow where the velocity is proportional to $1 - (r/r_0)^2$. This is a classical case and the corresponding surface-to-mean coolant temperature drop can readily be shown to be $11/24 \times r_0 q_0/\bar{k}^1$

i.e.

$$\bar{t}_a = \frac{r_0 q_0}{k}\left[\frac{s}{r_0} + \frac{11}{24}\frac{k}{\bar{k}^1}\right]$$

or

$$S_0 = \frac{s}{r_0} + \frac{11}{24}\frac{k}{\bar{k}^1}. \quad (5)$$

For laminar flow $s/r_0 \to 0$ and $k/\bar{k}^1 \to 1$, but equation (5) is expressed in the same form as equation (4) to show the upper limit of S_0 for flows which are just turbulent.

Equations (3)–(5) give

$$S_1 - S_0 = D\frac{k}{\bar{k}^1} \quad (6)$$

where D is function of Reynolds number and lies between 13/24 for laminar flow and 3/4 for very high values of Reynolds number. The actual value will be discussed later, and

$$S_1 - S_n = \frac{n-1}{n}\frac{k}{\bar{k}^1}. \quad (7)$$

Equations (6) and (7) enable $S_1, S_2, \ldots$, etc. to be readily calculated if the value of $k/\bar{k}^1$ can be determined, since $S_0 = 2/Nu$ can be considered to be known. It should be emphasised that these expressions do not require any information about the boundary layer and would have been the same if much more elaborate assumptions had been made about the temperature distribution in the fluid near the wall. It is for this reason that accurate simple expressions can be deduced for the *differences* between the non-dimensional temperatures and these will apply to rough as well as smooth surfaces.

The mean eddy diffusivity of heat, $\bar{\epsilon}_h$, is defined by,

$$\bar{k}^1 = \rho\sigma\bar{\epsilon}_h + k$$

or

$$\frac{\bar{k}^1}{k} = Pr\frac{\bar{\epsilon}_m}{\nu}\frac{\bar{\epsilon}_h}{\bar{\epsilon}_m} + 1. \quad (8)$$

The terms in this equation are:

(1) Pr is the Prandtl number of the fluid.

(2) ϵ_m/ν is the ratio of the eddy diffusivity of momentum to the molecular diffusivity of momentum. Many expressions have been proposed for its distribution over the turbulent core (e.g. see [1] and [3]) all suggest that the diffusivity is substantially uniform over the core and has an average value of

$$\bar{\epsilon}_m = \frac{u^* r_0}{A}. \quad (9)$$

Where u^* is the friction velocity defined by

$$\frac{\bar{u}}{u^*} = \bar{u}^+ = \sqrt{\frac{2}{f}} \quad (10)$$

$= 6{\cdot}60\ Re^{0{\cdot}1}$ for a smooth circular tube with turbulent flow and A is a constant. Reference [3] suggests a lower limit of 10 for A and shows that this is in good agreement with some experimental data.

$$\frac{\bar{\epsilon}_m}{\nu} = \frac{u^* r_0}{\nu A} = \frac{Re}{2\bar{u}^+ A}. \quad (11)$$

(3) $\bar{\epsilon}_h/\bar{\epsilon}_m$ is the ratio of the eddy diffusivity of heat to the eddy diffusivity of momentum. Theoretical estimates of this ratio have been made by several workers. It is usually argued that for $Pr \geqslant 1$ this ratio is one, but for lower values of Pr, particularly at low values of Re, conduction within each eddy reduces the eddy diffusivity of heat. A typical solution due to Jenkins [4] is

$$\frac{\bar{\epsilon}_h}{\bar{\epsilon}_m} = \frac{\frac{2}{15}\frac{\bar{\epsilon}_m}{\nu} + 1}{\frac{2}{15}\frac{\bar{\epsilon}_m}{\nu} + \frac{1}{Pr}}. \tag{12}$$

Substituting equations (11) and (12) in equation (8) gives

$$\frac{\bar{k}_1}{k} = \frac{PrRe}{2\bar{u}^+ A}\,\frac{\frac{Pr\,Re}{15\bar{u}^+ A} + Pr}{\frac{Pr\,Re}{15\bar{u}^+ A} + 1} + 1 \tag{13}$$

as an expression for $\bar{k}^1/k$ if the appropriate value for the empirical constant A is known.

COMPARISON WITH REYNOLDS CALCULATIONS

Reynolds [1] has calculated $S_n (n = 0$ to $n = 5)$ for a wide range of values of Prandtl and

Table 1. Reynolds results

Pr	Re	$\frac{\bar{k}'}{k}$	$\frac{\bar{k}'}{k}(S_1 - S_0)$	$\frac{n}{n-1}\frac{\bar{k}'}{k}(S_1 - S_n)$			
				$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	10^4	1	0·682	1	1	1	1
	3×10^4	1	0·698	1	1	1	1
	10^5	1	0·707	1	1	1	1
	3×10^5	1	0·712	1	1	1	1
	10^6	1	0·717	1	1	1	1
0·001	10^4	1·00008	0·682	1·000	1·000	1·000	1·000
	3×10^4	1·00045	0·689	1·000	1·000	1·000	1·000
	10^5	1·00356	0·708	1·004	1·003	1·002	1·002
	3×10^5	1·0240	0·709	0·990	0·989	0·992	0·993
	10^6	1·1885	0·778	1·032	1·037	1·040	1·048
0·003	10^4	1·00068	0·683	0·999	1·000	1·000	1·000
	3×10^4	1·00401	0·694	0·996	0·997	0·998	0·996
	10^5	1·0307	0·697	0·977	0·978	0·978	0·980
	3×10^5	1·1965	0·700	0·947	0·956	0·960	0·962
	10^6	2·340	0·742	0·910	0·948	0·982	0·960
0·01	10^4	1·0074	0·685	0·995	0·995	0·997	0·997
	3×10^4	1·0431	0·696	0·978	0·982	0·983	0·973
	10^5	1·303	0·664	0·875	0·886	0·897	0·903
	3×10^5	2·655	0·685	0·865	0·890	0·905	0·921
	10^6	9·45	0·902	0·985	1·013	1·040	1·062
0·03	10^4	1·064	0·675	0·975	0·993	0·965	0·970
	3×10^4	1·339	0·642	0·847	0·859	0·868	0·875
	10^5	3·04	0·674	0·815	0·835	0·851	0·867
	3×10^5	9·73	0·808	0·903	0·925	0·945	0·968
	10^6	35·3	1·012	0·940	0·963	0·983	1·005
0·7	10^4	14·55	0·842	0·90	0·92	0·96	0·98
	3×10^4	39·9	0·827	0·89	0·92	0·94	0·96
	10^5	119·5	0·813	0·93	0·95	0·97	0·98
	3×10^5	322	0·825	0·92	0·95	0·97	0·98
	10^6	958	0·969	0·94	0·95	0·97	0·99

Pr	Re	$\frac{\bar{k}'}{k}$	$\frac{\bar{k}'}{k}(S_1 - S_0)$	$\frac{n}{n-1}\frac{\bar{k}'}{k}(S_1 - S_n)$			
				$n = 2$	$n = 3$	$n = 4$	$n = 5$
3	10^4	65·2	0·80	0·90	0·93	0·96	0·99
	3×10^4	176	0·78	0·95	0·95	0·96	0·99
	10^5	518	0·64	0·92	0·94	0·96	0·97
	3×10^5	1390	0·72	0·92	0·94	0·97	0·97
	10^6	4120	0·87	0·92	0·93	0·96	0·97
10	10^4	218	0·83	0·92	0·92	0·96	0·98
	3×10^4	586	0·73	0·91	0·94	0·96	0·98
	10^5	1730	0·55	0·90	0·94	0·95	0·98
	3×10^5	4640	0·56	0·93	0·90	0·93	0·93
	10^6	13700	0·75	0·90	0·93	0·95	0·96
30	10^4	652	0·6	0·9	1·0	1·0	1·0
	3×10^4	1760	0·5	0·9	1·0	1·0	1·0
	10^5	5180	0·3	1·0	0·9	1·0	1·0
	3×10^5	13900	0·3	0·9	0·9	1·0	1·0
	10^6	41200	0·5	0·9	0·9	0·9	1·0
100	10^4	$2·18 \times 10^3$	0·4	0·9	1·0	1·0	1·0
	3×10^4	$5·86 \times 10^3$	−0·1	0·9	1·0	1·0	1·0
	10^5	$17·3 \times 10^3$	−0·3	1·0	1·0	0·9	1·0
	3×10^5	$46·4 \times 10^3$	−0·5	0·9	1·0	1·0	1·0
	10^6	137×10^3	−0·1	0·8	0·8	0·9	0·9
1000	10^4	$2·18 \times 10^4$	−3	0·9	1·0	1·2	1·1
	3×10^4	$5·86 \times 10^4$	−4	1·2	1·7	1·6	1·5
	10^5	$17·3 \times 10^4$	−7	0·7	1·0	0·9	0·9
	3×10^5	$46·4 \times 10^4$	−8	0·9	0·9	0·6	0·6
	10^6	137×10^4	−6	0·8	1·1	0·9	0·9

Reynolds numbers. The assumed distribution of eddy diffusivity of momentum is equivalent to a value of A of about 16. For the value of $\bar{\epsilon}_h/\bar{\epsilon}_n$, Reynolds uses Jenkins' expression multiplied by 1·15 for low values of Prandtl number and 1·15 for high values i.e.

$$\frac{\bar{k}_1}{k} \simeq \frac{1{\cdot}15\, Re\, Pr}{32\bar{u}^+} \frac{Re\, Pr + Pr}{\dfrac{Re\, Pr}{240\bar{u}^+} + 1} + 1, \quad Pr \leqslant 0{\cdot}7 \quad (14)$$

$$\frac{\bar{k}^1}{k} \simeq \frac{1{\cdot}15\, Re\, Pr}{32\bar{u}^+} + 1, \quad Pr \geqslant 3 \quad (15)$$

Table 1 gives values of $\bar{k}^1/k$ for these expressions and values of $(\bar{k}^1/k)$ $(S_1 - S_0)$ and $(\bar{k}^1/k)$ $[n/(n-1)](S_1 - S_n)$ for $n = 2, 3, 4$ and 5 using the values of S_n tabulated in [1]. It can be seen that $[n/(n-1)](\bar{k}^1/k)(S_1 - S_n)$ is very nearly equal to 1 over the whole ranges of Prandtl and Reynolds numbers. This shows the approximate formula equation (7) agrees with Reynolds calculations to just about the accuracy to which these differences were computed.

For $Pr = 0$, the values of $(\bar{k}^1/k)(S_1 - S_0)$ are the values of D in equation (6) and are shown plotted as a function of Reynolds number in Fig. 1. For turbulent flows it is usually sufficiently accurate to use $D = 0{\cdot}70$. If there were perfect agreement between Reynolds calculations and equation (6), the variation of $(\bar{k}^1/k)(S_1 - S_0)$ with Reynolds number would be exactly repeated for all values of Prandtl number. This is approximately true for the smaller values of Prandtl number; as the Prandtl number increases the discrepancy becomes larger and some negative

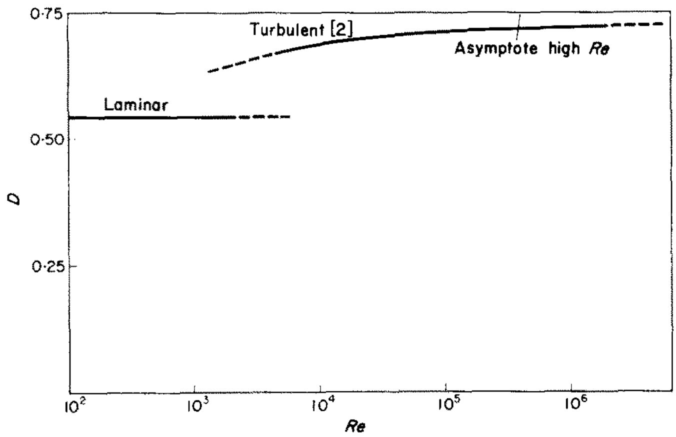

FIG. 1. D as a function of Reynolds number.

values are given. These discrepancies probably arise because the values of S_0 have been taken from a different computation which used similar, but not identical, expressions for the distribution of the eddy diffusivities of momentum and heat. For the higher values of Prandtl number, $S_1 - S_0$ is only a small proportion of S_1 or S_0 and very small discrepancies in the method of calculation could give relatively large errors in $S_1 - S_0$. The approximate expression, equation (6), is probably more accurate therefore than the values given in [1].

It is concluded that if S_0, i.e. Nu, is known, S_n may be obtained from equations (6) and (7) with sufficient accuracy for practical calculations. The accuracy is limited by the accuracy with which $\bar{k}^1/k$ can be specified, the particular values used in [1] are subject to considerable uncertainty (see below). The difference between the effort required to use equations (6) and (7) and the effort required to carry out the numerical calculation may be noted.

COMPARISON WITH BLACK AND SPARROW'S EXPERIMENTS

Black and Sparrow [2] investigated the non-uniform heat transfer from the circumference of a circular tube to air flowing through it. The heat is generated non-uniformly by using a tube with a wall thickness of $b(1 + B \cos\theta)$ and passing an electric current along it. The local heat generation rate per unit volume of tube material is proportional to the local tube material resistivity, since the axial voltage gradient does not vary around the tube. If to a first approximation the wall temperature variation about its mean value is $t_1 \cos\theta$, the local material resistivity, ω is given by

$$\frac{\omega}{\omega_0} = 1 + \frac{t_1}{\omega_0}\frac{\partial\omega}{\partial t}\cos\theta. \tag{16}$$

Where ω_0 is the resistivity at the mean circumferential wall temperature.

In the experiments $\dfrac{1}{\omega_0}\dfrac{\partial\omega}{\partial t} \simeq 0{\cdot}0011/^\circ\mathrm{C}$

$$t_1 \simeq 9^\circ\mathrm{C}$$

$$\frac{\text{Heat generation rate}}{\text{Mean heat generation rate}} = 1 + C\cos\theta$$

$$= \frac{1 + B\cos\theta}{1 + \dfrac{t_1}{\omega_0}\dfrac{\partial\omega}{\partial t}\cos\theta}$$

$\therefore B - C \simeq 0{\cdot}01.$

The results are presented as graphs of the measured inner wall temperature variation about the mean value divided by the mean wall-to-gas temperature difference (the outer wall temperatures are actually measured and a correction applied, but this is small, see Appendix) and graphs of the calculated wall heat flux divided by the mean wall heat flux. The heat flux variation is obtained by an elaborate solution to the problem of conduction in the tube wall. The local inner wall heat flux is assumed to be the difference between the local generation rate and the neat heat conducted to that point round the tube wall due to the measured temperature distribution. This heat flux variation is probably overestimated for two reasons; first, the variation in heat generation is assumed to be B not C, i.e. resistivity changes are ignored, and secondly, although an allowance is made for the overall loss from the outer surface through the insulation, no allowance is made for the contribution of the insulation to the redistribution of the heat flux circumferentially.

An alternative analysis (see Appendix) shows that the variation of tube wall thickness will introduce higher harmonics of the temperature and heat flux variations, but to a very good approximation, for the ratio of the tube wall thickness to tube radius actually used, the first harmonic amplitudes are given by:

$$\frac{q_1}{q_0} = \frac{C}{1 + N(S_1/S_0)} \tag{17}$$

and

$$\frac{t_1}{\bar{t}_a} = \frac{C(S_1/S_0)}{1 + N(S_1/S_0)}. \tag{18}$$

Since for gas cooling $Pr \sim 1$, $\bar{k}^1/k \gg 1$ (see Table 1)

$$\therefore \; \frac{\bar{k}^1}{k} \simeq Pr \frac{\bar{\epsilon}_m}{\nu} \frac{\bar{\epsilon}_h}{\bar{\epsilon}_m}. \tag{19}$$

Using equation (4), (6), (11) and (19)

$$\frac{S_1}{S_0} = 1 + \frac{S_1 - S_0}{S_0} = 1 + \frac{AD}{\bar{t}_a^+(\bar{\epsilon}_h/\bar{\epsilon}_m)} \tag{20}$$

where

$$\bar{t}_a^+ = \frac{Pr\,Re}{\bar{u}^+\,Nu} \tag{21}$$

is the non dimensional surface-mean coolant temperature drop [3]. If the following numerical values are assumed

$r_0 = 14{\cdot}89$ mm, $b = 0{\cdot}84$ mm
$k = 0{\cdot}0278$ $W/\mathrm{m}^\circ\mathrm{C}$, $k_s = 16{\cdot}5$ $W/\mathrm{m}^\circ\mathrm{C}$
$Pr = 0{\cdot}725$
$Nu = 0{\cdot}95 \times 0{\cdot}023\, Re^{0{\cdot}8}\, Pr^{0{\cdot}4}$ (Fig. 12 [2])
$\quad = 0{\cdot}0192\, Re^{0{\cdot}8}$
$N = 3{,}380\, Re^{-0{\cdot}8}$, using equation (27)
$\bar{t}_a^+ = 5{\cdot}72\, Re^{0{\cdot}1}$, using equations (10) and (21)
$C = 0{\cdot}50$
D is taken from Fig. 1.

q_1/q_0 and $t_1/\bar{t}_a$ can be determined as functions of Re for various values of $A\epsilon_m/\epsilon_h$ using equations (17), (18) and (20). They are shown as curves in Fig. 2 together with the values taken from the figures in [2]. The experimental values of $t_1/\bar{t}_a$

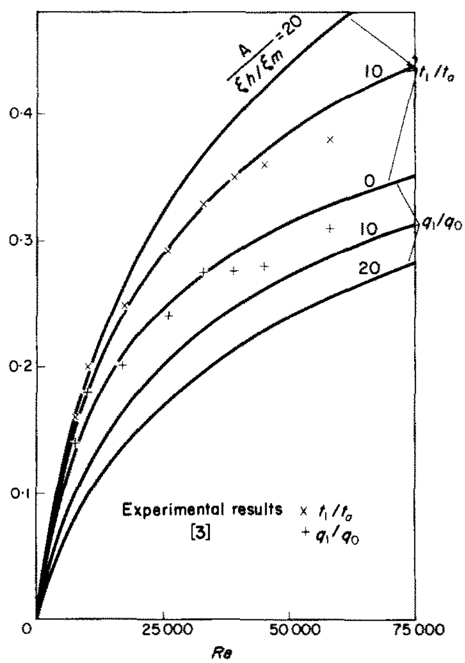

FIG. 2. Amplitudes of first harmonics.

are in good agreement with the theoretical line for $A\epsilon_m/\epsilon_h = 10$, remembering the probable experimental errors and the possible errors in the numerical data used for the theoretical estimates, e.g. C should be reduced to allow for resistivity variations. As explained above, Black and Sparrows' estimates of q_1/q_0 are probably too large. Allowance for resistivity variations would probably reduce each value by a constant of about 0·01. Allowance for additional damping by the lagging would reduce all the values, with relatively greater reductions at the lower values of Reynolds number.

The conclusion is that the measured temperature variations are consistent with a value of $A\epsilon_m/\epsilon_h$ of 10 and the corrected values of q_1/q_0 would support this. The circumferential variation of Nusselt number is therefore greater than given by Black and Sparrow and in better agreement with the predictions of Sparrow and Lin [5]. This discrepancy could not be satisfactorily explained in [2].

DISCUSSION AND CONCLUSIONS

The approximate formulae [equations (6) and (7)] for the non-dimensional amplitudes of the harmonics of wall temperature variation, S_n ($n \geqslant 1$), have been shown to be in good agreement with the numerical calculations of Reynolds [1]. They can therefore be used as working formulae for practical calculations. Their accuracy is limited by the accuracy with which it is possible to specify the ratio of the mean effective eddy conductivity of the fluid to its molecular conductivity, $\bar{k}^1/k$. This also applies to the detailed numerical calculations.

An analysis of the experimental results of Black and Sparrow [2] suggests a mean level of eddy conductivity corresponding to $A\epsilon_m/\epsilon_h = 10$ or, since $\epsilon_h/\epsilon_m \doteq 1$ for their experimental conditions,

$A = 10$ i.e. from equation (9)

$$\bar{\epsilon}_m = \frac{u^* r_0}{10}. \qquad (22)$$

This is close to the value suggested in [3], but a rather larger diffusivity than used in many calculations, e.g. [1].

Equations (11) and (12) give for $Pr < 1$,

$$\frac{\epsilon_h}{\epsilon_m} = \frac{\dfrac{PrRe}{15\bar{u}^+ A} + Pr}{\dfrac{PrRe}{15\bar{u}^+ A} + 1}. \qquad (23)$$

For turbulent flow $\bar{u}^+ \sim 15$–25, [equation (10)].

$\dfrac{\epsilon_h}{\epsilon_m}$ can be assumed to be 1 if $\dfrac{PrRe}{15\bar{u}^+ A} \gg 1$

i.e. if $Pr\,Re$ = Péclet number $> 10^4$

(this must be turbulent flow since the Jenkins' formula only applied for $Pr < 1$).

Equation (13) then reduces to

$$\frac{\bar{k}^1}{k} = \frac{Pr\,Re}{20\bar{u}^+}.$$

From Fig. 1, $D \simeq 0{\cdot}70$, equations (4), (6) and (21) give

$$\frac{S_1 - S_0}{S_0} = \frac{7}{\bar{t}_a^+}, \qquad (24)$$

equations (4), (7) and (21) give

$$\frac{S_1 - S_n}{S_0} = \frac{n-1}{n}\frac{10}{\bar{t}_a^+} \qquad (25)$$

$$\therefore \frac{t_n}{\bar{t}_a} = \frac{S_n}{S_0}\frac{q_n}{q_0}$$

$$= \left(1 + \frac{S_1 - S_0}{S_0} - \frac{S_1 - S_n}{S_0}\right)\frac{q_n}{q_0}$$

$$= \left(1 + \frac{\dfrac{10}{n} - 3}{\bar{t}_a^+}\right)\frac{q_n}{q_0}. \qquad (26)$$

For turbulent flow with Péclet numbers greater than 10^4, equation (26) provides a simple relationship between the amplitudes of the harmonics of the temperature and heat flux variations to the accuracy of the available experimental data.

Define $h_0 = q_0/\bar{t}_a$ = mean heat-transfer coefficient of the channel and $h_n = q_n/t_n$ = amplitude of nth harmonic of heat-transfer coefficient.

For turbulent flow of gases ($Pr \simeq 0{\cdot}7$), $\bar{t}_a^+$ will be typically 15 for smooth surfaces and 7 for very rough surfaces. Substituting in equation (26) gives

	Smooth	Rough
$\frac{h_0}{h_1} = 1 + \frac{7}{\bar{t}_a^+}$	1·47	2·00
$\frac{h_0}{h_2} = 1 + \frac{2}{\bar{t}_a^+}$	1·13	1·29
$\frac{h_0}{h_3} = 1 + \frac{1}{3\bar{t}_a^+}$	1·02	1·05
$\frac{h_0}{h_4} = 1 - \frac{1}{2\bar{t}_a^+}$	0·97	0·93
	etc.	

These values can be compared with two simpler assumptions which have been suggested. Firstly that the local heat-transfer coefficient is constant around the circumference, i.e. $h_0/h_n = 1$, which over-estimates h_1, h_2 *and* h_3, i.e. is optimistic in under-estimating the magnitude of the temperature variations. Secondly that the local heat-transfer coefficient is inversely proportional to the local heat flux, i.e. $h_0/h_n = 2$ for small variations from the mean. This under-estimates h_n except for the first harmonic with a very rough surface. Usually only the first harmonic is of practical interest and a useful working rule for smooth tubes to the accuracy of the available experimental data is $h_1/h_0 = 2/3$ for turbulent flow in gases.

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APPENDIX

The Temperature Distribution in a Heated Tube of Varying Wall Thickness

Consider a tube of internal radius r_0 and wall thickness $b\,(1 + B\cos\theta)$. If it is heated by passing an electric current through it, the local rate of heat supply per unit inside surface area will be $q_0\,(1 + C\cos\theta)$ where $C = B$ if the material resistivity is constant around the circumference, but in general the resistivity will vary with temperature. There will be a parabolic distribution of temperature in a radial direction with a slope of $-q_0/k_s$ at the inner surface and zero at the outer surface assuming no loss there.

$$\therefore \text{Temperature drop across thickness} = \frac{q_0 b}{2k_s}.$$

At $\theta = \pi/2$, temperature difference between surface and coolant $= \bar{t}_a$

$$= \frac{2q_0 r_0}{Nuk}$$

$$\therefore \frac{\text{Mean temperature drop across thickness}}{\bar{t}_a} = Nu\frac{b}{4r_0}\frac{k}{k_s} \simeq \frac{1}{2N}\left(\frac{b}{r_0}\right)^2.$$

Where

$$N = \frac{2bk_s}{Nu[r_0 + (b/2)]\,k} = \frac{\bar{t}_a bk_s}{q_0 r_0[r_0 + (b/2)]}. \tag{27}$$

For Black and Sparrow's [3] experiments

$$\frac{\text{Mean temperature drop across thickness}}{\bar{t}_a} \simeq \frac{1}{1000}.$$

It is possible therefore to neglect radial variations in temperature in determining the equation for the redistribution of heat flux around the tube by conduction. It is also only necessary to consider solutions which are symmetrical about the line $\theta = 0$.

$\therefore$ Local rate of heat supply per unit inside surface area $=$ $q_0(1 + C\cos\theta) = q_0 + q_1\cos\theta + q_2\cos 2\theta + \ldots + q_n\cos n\theta$

$$+ \frac{1}{[r_0 + (bB/2)\cos\theta]}\frac{\partial}{\partial\theta}\left[-\frac{k_s b(1 + B\cos\theta)}{[r_0 + (b/2)]}\frac{\partial t}{\partial\theta}\right]$$

$$r_0 - \frac{bB}{2}\cos\theta \simeq r_0 \text{ since } \cos\theta \leqslant 1, \quad b \ll r \text{ and } B \ll 1$$

$$\therefore C\cos\theta = \frac{q_1}{q_0}\cos\theta + \frac{q_2}{q_0}\cos 2\theta + \ldots + \frac{q_n}{q_0}\cos n\theta$$

$$- N\left[(1 + B\cos\theta)\frac{\partial^2}{\partial\theta^2}\left(\frac{t}{\bar{t}_a}\right) \cdot B\sin\theta\frac{\partial}{\partial\theta}\left(\frac{t}{\bar{t}_a}\right)\right].$$

From equations (2) and (3)

$$S_0\frac{\partial}{\partial\theta}\left(\frac{t}{\bar{t}_a}\right) = -S_1\frac{q_1}{q_0}\sin\theta - 2S_2\frac{q_2}{q_0}\sin 2\theta \ldots - nS_n\frac{q_n}{q_0}\sin n\theta$$

$$S_0\frac{\partial^2}{\partial\theta^2}\left(\frac{t}{\bar{t}_a}\right) = -S_1\frac{q_1}{q_0}\cos\theta - 4S_2\frac{q_2}{q_0}\cos 2\theta \ldots - n^2S_n\frac{q_n}{q_0}\cos n\theta.$$

Substituting these values gives

$$0 = \left[-CS_0 + (S_0 + NS_1)\frac{q_1}{q_0} + NBS_2\frac{q_2}{q_0}\right]\cos\theta$$

$$+ \left[NBS_1\frac{q_1}{q_0} + (S_0 + 4NS_2)\frac{q_2}{q_0} + 3NBS_3\frac{q_3}{q_0}\right]\cos 2\theta$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \left[\frac{n(n-1)}{2}NBS_{n-1}\frac{q_{n-1}}{q_0} + (S_0 + n^2NS_n)\frac{q_n}{q_0}\right.$$

$$\left. + \frac{n(n+1)}{2}NBS_{n+1}\frac{q_{n+1}}{q_0}\right]\cos n\theta.$$

Each square bracket must be zero and a solution for any number of harmonics can be obtained by assuming all the higher harmonics are negligible.

For large values of n, $(q_n/q_{n-1}) \rightarrow -(B/2)$ i.e. the amplitudes of the harmonics of heat flux usually decreases rapidly. Assuming that q_2 is negligible

$$\frac{q_1}{q_0} = \frac{CS_0}{S_0 + NS_1}. \tag{17}$$

Assuming that q_3 is negligible

$$\frac{q_2}{q_1} = -\frac{NBS_1}{S_0 + 4NS_2}$$

then

$$q_1 = \frac{CS_0}{S_0 + NS_1 - \dfrac{N^2B^2S_1S_2}{S_0 + 4NS_2}}$$

$$\rightarrow \frac{CS_0}{S_0 + NS_1} \text{ if } N \ll 1 \quad \text{or} \quad B \ll 1$$

$$\rightarrow \frac{CS_0}{S_0 + NS_1\left(1 - \dfrac{B^2}{4}\right)} \text{ if } N \gg 1.$$

$\therefore$ Equation (17) is a good approximation for q_1/q_0 for most practical purposes

$$\frac{t_1}{\bar{t}_a} = \frac{S_1}{S_0}\frac{q_1}{q_0} \simeq \frac{CS_1}{S_0 + NS_1}. \tag{18}$$

TRANSFERT THERMIQUE DE CONVECTION FORCÉE DANS UN TUBE CIRCULAIRE AVEC FLUX THERMIQUE CIRCONFÉRENTIEL NON UNIFORME

Résumé—On obtient une solution analytique basée sur une hypothèse de distribution simple de la conductivité thermique effective du fluide, pour les variations de température sur la circonférence d'un tube circulaire, variations résultant d'un flux thermique non uniforme à la paroi. Les résultats sont en bon accord avec une solution numérique complexe déjà existante, et dans certains cas sont probablement plus précis. L'exactitude des deux solutions est principalement déterminée par la précision avec laquelle est connue la diffusivité par turbulence effective de la chaleur dans le fluide.

Des résultats expérimentaux publiés pour un tube chauffé non uniformément sont utilisés pour établir la meilleure valeur. On propose une formule simple pour calculer les variations de température autour de tubes dans des cas pratiques.

WÄRMEÜBERTRAGUNG DURCH ERZWUNGENE KONVEKTION IN EINEM KREISFÖRMIGEN ROHR MIT NICHT GLEICHBLEIBENDEM WÄRMSTROM LÄNGS DES UMFANGS

Zusammenfassung—Eine analytische Lösung, beruhend auf einer einfach postulierten Verteilung der effektiven Wärmeleitfähigkeit des Fluids, wurde für die Temperaturschwankungen längs des Umfangs eines kreisförmigen Rohres, die von einem nicht gleichmässigen Wandwärmestrom herrühren, abgeleitet. Die Ergebnisse sind in guter Übereinstimmung mit einer existierenden, komplizierten numerischen Lösung und in einigen Fällen wahrscheinlich genauer.

Die Genauigkeit der beiden Lösungen wird hauptsächlich durch die Genauigkeit bestimmt, mit der der effektive Scheindiffusionskoeffizient der Wärme in dem Fluid bekannt ist. Veröffentlichte experimentelle Werte für nicht gleichmässig beheizte Rohre wurden benutzt, um den besten Wert festzustellen. Eine einfache Formel zur Berechnung von Temperaturschwankungen rings um Rohre in praktischen Fällen wird vorgeschlagen.

ТЕПЛООБМЕН В КРУГЛОЙ ТРУБЕ С НЕОДНОРОДНЫМ ТЕПЛОВЫМ ПОТОКОМ ПО ПЕРИМЕТРУ ПРИ ВЫНУЖДЕННОЙ КОНВЕКЦИИ

Аннотация—Получено аналитическое решение, основанное на простом постулированном распределении эффективной теплопроводности жидкости, для температурных изменений по окружности круглой трубы, вызванных неоднородным потоком тепла на стенке. Результаты находятся в хорошем соответствии с существующим сложным численным решением, а в некоторых случаях, вероятно, более точны. Точность обоих решений, в основном, определяется точностью, известной для эффективной турбулентной температуропроводности в жидкости. Опубликованные экспериментальные данные для неоднородно нагретой трубы используются для установления наилучшего значения. Предложена простая формула расчёта температурных изменений на практике.